ON THE OPTIMIZATION OF CONTROLLED SYSTEMS WITH RANDOM PARAMETERS

(K OPTIMIZATSII UPRAVLIAEMYKH SISTEM SO SLUCHAINYMI PARAMETRAMI)

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At the present time there is a number of works devoted to the investigation of optimum dynamical systems. In [1 to 4], for example, the question of selecting the system operator to which the greatest accuracy of transforming the input action corresponds, is considered. In [5 to 8] and others, the problem of selecting the operation law of a control which will guarantee the best approximation of the perturbed motion of an object to a prescribed program, is solved. The latter is usually found from the condition of achieving the extremum value of some characteristic of the object under the so-called rated motion conditions. The works [9 to 11] as well as those metioned in their bibliography are devoted to looking for optimum, in this sense, flight conditions of flying vehicles.

The accuracy of realizing programed motion depends not only on the properties of controls but also on the properties of the programed motion itself and on the rated parameters of the system. It may sometimes appear to be expedient to raise the accuracy of the system motion on the account of change in its programed motion and parameters. Such a change leads to a reduction in the level of the characteristic of the object whose extremum is to be achieved but the accuracy in its performance is raised.

The present study is devoted to seeking the control functions and parameters which are optimum in this last sense.

1. Let the equations

$$X_i^{\cdot} = f_i(t, X_1, \ldots, X_n, U_1, \ldots, U_r, A_1, \ldots, A_q, P_1, \ldots, P_\nu) \qquad (i = 1, \ldots, n) \tag{1.1}$$

with the initial conditions

$$X_i(t_0) = X_{i0} \qquad (i = 1, \ldots, n) \tag{1.2}$$

describe the system motion.

Here t is the time, X_i $(i = 1, \ldots, n)$ the phase coordinates which are continuous functions of time; U_j $(j = 1, \ldots, r)$ are the random control functions of time with the canonical expansion [1] of the form

$$U_j(t) = u_j(t) + \sum_{k=1}^{m} V_{jk} \xi_{jk}(t)$$

Here $u_j(t)$ is the mathematical expectation of the function $U_j(t)$, $\xi_{jk}(t)$ is a given coordinate function, $V_{jk} \equiv V_i$ $(i = 1, \ldots, mr)$ are uncorrelated random variables with mathematical expectations zero and given variances D_i^v, X_{i0}, A_i, P_i are random variables with the mathematical expectations x_{i0}, a_i, p_i and the given variances D_{i0}^x, D_i^a, D_i^p.

The parameters x_{i0}, p_i, t_0 are given and the $a_1, \ldots, a_q$ are among the control factors. The random variables

$$V_i, \quad X_{i0}^\circ = X_{i0} - x_{i0}, \quad A_i^\circ = A_i - a_i, \quad P_i^\circ = P_i - p_i$$

are considered uncorrelated. The prescribed functions $f_1, \ldots, f_n$ in (1.1) are assumed to be contunous together with their derivatives to third order, inclusively.

Because of the conditions of the formulated problem, the coordinates $X_i(t)$ $(i = 1, \ldots, n)$ will be random functions of time with the mathematical expectations $x_i(t)$ $(i = 1, \ldots, n)$ and variances $D_i^x(t)$ $(i = 1, \ldots, n)$.

Let us consider V_i, X_{i0}°, A_i°, P_i° to be so small that linearization of (1.1) with respect to these parameters and of the function

$$X_i^\circ(t) = X_i(t) - x_i(t)$$

is possible.

Then we obtain for the determination of $x_i(t)$ and $X_i^\circ(t)$ [1]

$$x_i^\cdot = f_i(t, x_1, \ldots, x_n, u_1, \ldots, u_r, a_1, \ldots, a_q, p_1, \ldots, p_v), \qquad x_i(t_0) = x_{i0} \qquad (i = 1, \ldots, n) \tag{1.3}$$

$$X_i^{\circ\cdot} = \sum_{k=1}^{n} \alpha_{ik} X_k^\circ + \sum_{k=1}^{mr} \beta_{ik} V_k + \sum_{k=1}^{q} \gamma_{ik} A_k^\circ + \sum_{k=1}^{v} \zeta_{ik} P_k^\circ, \qquad X_i(t_0) = X_{i0} \qquad (i = 1, \ldots, n) \tag{1.4}$$

Here the $\alpha_{ik}, \beta_{ik}, \gamma_{ik}, \zeta_{ik}$ are known functions of the parameters t, x_i, u_i, a_i and p_i.

Equations (1.3) describe the programed motion of the system, the mean realization of the random process $[X_i(t), U_j(t)]$.

Let us assume the existence of the set of controls

$$u_j(t) \ (j = 1, \ldots, r), \ a_j \qquad (j = 1, \ldots, q)$$

which may transform the system from the initial state x_{i0} into a final state characterized by the relations

$$\psi_j(x_{11}, \ldots x_{n1}, a_1, \ldots, a_q, p_1, \ldots, p_v) = 0, \qquad x_{i1} = x_i(t_1) \qquad (i = 1, \ldots, n; \ j = 1, \ldots, k, \ k \leqslant n + q) \tag{1.5}$$

Here t_1 the final value of the time, is a given number, the ψ_j are known continuous functions with continuous derivatives to third order, inclusively.

Equations (1.4) as well as the parameters

$$\Psi_j = \psi_j(X_{11}, \ldots, X_{n1}, A_1, \ldots, A_q, P_1, \ldots, P_v) \qquad (j = 1, \ldots, k)$$

which may be evaluated to second order accuracy by means of the formulas

$$\Psi_j = \sum_{i=1}^{n} \frac{\partial \psi_j}{\partial x_{i1}} X_{i1}^\circ + \sum_{i=1}^{q} \frac{\partial \psi_j}{\partial a_i} A_i^\circ + \sum_{i=1}^{v} \frac{\partial \psi_j}{\partial p_i} P_i^\circ \tag{1.6}$$

permit the finding of the deviation of realizing the random process from the programed state. As we see, this deviation depends not only on the random parameters but also on the characteristics of the programed motion.

It is natural to attempt to find the control $u_1, \ldots, u_r$, $a_1, \ldots, a_q$ to which the least level of scattering of the realization of the random process $[X_i(t)]$ and the random parameters Ψ_j $(j = 1,\ldots, k)$ will correspond. We shall designate the control satisfying such a constraint as optimum in the mean.

2. Let us consider that the functional

$$I = \int_{t_0}^{t_1} \Big(\sum_{i=1}^{n} b_i D_i^x + \sum_{i=1}^{r} m_i u_i^2 \Big) dt + \sum_{i=1}^{k} g_i D_i^\psi \tag{2.1}$$

characterizes the scattering level of the trajectories $[X_i(t)]$, of the parameters Ψ_j $(j = 1,\ldots, k)$ and the measure of the control effects.

Here b_i, m_i and g_i are non-negative weight constants, D_i^x and D_i^ψ the variances of the functions $X_i(t)$ and the parameters Ψ_j.

Then the problem of selecting the optimum, in the mean, control for a bounded level of the control effects $u_1,\ldots, u_r$ reduces to seeking the functions u_i, x_i, X_i° and the parameters a_i satisfying (1.3) and (1.5) and creating the minimum for the functional (2.1).

Let us transform this variational problem to a form more convenient for solution.

It is seen from (1.4) that $X_i^\circ(t)$ may be written as

$$X_i^\circ = \sum_{k=1}^{n} \eta_i^k X_{k0}^\circ + \sum_{k=1}^{mr} \eta_{vi}^k V_k + \sum_{k=1}^{q} \eta_{ai}^k A_k^\circ + \sum_{k=1}^{\nu} \eta_{pi}^k P_k^\circ \tag{2.2}$$

Here the η_i^k, η_{vi}^k, η_{ai}^k and η_{pi}^k are determined from Equations

$$H_i^k \equiv \frac{d\eta_i^k}{dt} - \sum_{j=1}^{n} \alpha_{ij}\eta_j^k = 0 \qquad (i, k = 1, \ldots, n)$$

$$H_{vi}^k \equiv \frac{d\eta_{vi}^k}{dt} - \sum_{j=1}^{n} \alpha_{ij}\eta_{vj}^k - \beta_{ik} = 0 \qquad (i = 1, \ldots, n;\ k = 1, \ldots, mr) \tag{2.3}$$

$$H_{ai}^k \equiv \frac{d\eta_{ai}^k}{dt} - \sum_{j=1}^{n} \alpha_{ij}\eta_{aj}^k - \gamma_{ik} = 0 \qquad (i = 1, \ldots, n;\ k = 1, \ldots, q)$$

$$H_{pi}^k \equiv \frac{d\eta_{pi}^k}{dt} - \sum_{j=1}^{n} \alpha_{ij}\eta_{pj}^k - \zeta_{ik} = 0 \qquad (i = 1, \ldots, n;\ k = 1, \ldots, \nu)$$

$$\eta_i^k(t_0) = 0 \quad (i \neq k), \qquad \eta_i^k(t_0) = 1 \quad (i = k), \qquad \eta_{vi}^k(t_0) = \eta_{ai}^k(t_0) = \eta_{pi}^k(t_0) = 0 \tag{2.4}$$

We have from (2.2)

$$D_i^x = \sum_{k=1}^{n} (\eta_i^k)^2 D_{k0}^x + \sum_{k=1}^{mr} (\eta_{vi}^k)^2 D_k^v + \sum_{k=1}^{q} (\eta_{ai}^k)^2 D_k^a + \sum_{k=1}^{\nu} (\eta_{pi}^k)^2 D_k^p$$

It follows from (1.6) and (2.2)

$$\Psi_i \equiv \sum_{k=1}^{n} c_{ik} X_{k0}^\circ + \sum_{k=1}^{mr} d_{ik} V_k + \sum_{k=1}^{q} l_{ik} A_k^\circ + \sum_{k=1}^{\nu} h_{ik} P_k^\circ$$

Here c_{ik}, d_{ik}, l_{ik}, h_{ik} are known functions of the final values of the variables x_i, η_i^k, η_{vi}^k, η_{ai}^k, η_{pi}^k and the parameters a_i and p_i. Hence, we have

$$D_i^{\psi} = \sum_{k=1}^{n} c_{ik}^2 D_{k0}^x + \sum_{k=1}^{mr} d_{ik}^2 D_k^v + \sum_{k=1}^{q} l_{ik}^2 D_k^a + \sum_{k=1}^{\nu} h_{ik}^2 D_k^p$$

Taking account of the obtained expressions for D_i^x, D_i^{ψ}, let us write the functional (2.1) as

$$I = \int_{t_0}^{t_1} F dt + \Phi \tag{2.5}$$

Here F is a known function of the variables $\eta_i^k, \eta_{vi}^k, \eta_{ai}^k, \eta_{pi}^k, u_i$ and the parameters $b_i, m_i, D_{i0}^x, D_i^v, D_i^a, D_i^p$ and Φ is a known function of the final values of the variables $x_i, \eta_i^k, \eta_{vi}^k, \eta_{ai}^k, \eta_{pi}^k$ and the parameters $g_i, a_i, p_i, D_{i0}^x, D_i^v, D_i^a, D_i^p$.

The problem under consideration may be formulated as a variational problem to determine the functions $u_i, x_i, \eta_i^k, \eta_{vi}^k, \eta_{ai}^k, \eta_{pi}^k$ and the parameters a_i satisfying Equations (1.3) to (1.5), (2.3), (2.4) and creating for the minimum for the functional (2.5).

3. Let us represent the system of necessary conditions as follows [12, 13]:

$$\frac{\partial F^*}{\partial x_i} - \frac{d}{dt}\frac{\partial F^*}{\partial \dot{x}_i} = 0 \quad (i = 1, \ldots, n), \qquad \frac{\partial F^*}{\partial \eta_{ai}^k} - \frac{d}{dt}\frac{\partial F^*}{\partial \dot{\eta}_{ai}^k} = 0 \quad \begin{pmatrix} i = 1, \ldots, n; \\ k = 1, \ldots, q \end{pmatrix}$$

$$\frac{\partial F^*}{\partial \eta_i^k} - \frac{d}{dt}\frac{\partial F^*}{\partial \dot{\eta}_i^k} = 0 \quad (i, k = 1, \ldots, n), \qquad \frac{\partial F^*}{\partial \eta_{pi}^k} - \frac{d}{dt}\frac{\partial F^*}{\partial \dot{\eta}_{pi}^k} = 0 \quad \begin{pmatrix} i = 1, \ldots, n; \\ k = 1, \ldots, \nu \end{pmatrix}$$

$$\frac{\partial F^*}{\partial \eta_{vi}^k} - \frac{d}{dt}\frac{\partial F^*}{\partial \dot{\eta}_{vi}^k} = 0 \quad \begin{pmatrix} i = 1, \ldots, n; \\ k = 1, \ldots, mr \end{pmatrix}, \qquad \frac{\partial F^*}{\partial u_i} = 0 \quad (i = 1, \ldots, r)$$

$$\frac{\partial \Phi^*}{\partial a_i} + \int_{t_0}^{t_1} \frac{\partial F^*}{\partial a_i} dt = 0 \quad (i = 1, \ldots, q), \qquad \lambda_i|_{t_1} + \frac{\partial \Phi^*}{\partial x_{i1}} = 0 \quad (i = 1, \ldots, n) \tag{3.1}$$

$$\left[\lambda_i^k + \frac{\partial \Phi^*}{\partial \eta_i^k}\right]_{t_1} = 0 \quad \begin{pmatrix} i = 1, \ldots, n; \\ k = 1, \ldots, n \end{pmatrix}, \qquad \left[\lambda_{ai}^k + \frac{\partial \Phi^*}{\partial \eta_{ai}^k}\right]_{t_1} = 0 \quad \begin{pmatrix} i = 1, \ldots, n; \\ k = 1, \ldots, q \end{pmatrix}$$

$$\left[\lambda_{vi}^k + \frac{\partial \Phi^*}{\partial \eta_{vi}^k}\right]_{t_1} = 0 \quad \begin{pmatrix} i = 1, \ldots, n; \\ k = 1, \ldots, mr \end{pmatrix}, \qquad \left[\lambda_{pi}^k + \frac{\partial \Phi^*}{\partial \eta_{pi}^k}\right]_{t_1} = 0 \quad \begin{pmatrix} i = 1, \ldots, n \\ k = 1, \ldots, \nu \end{pmatrix} \tag{3.2}$$

$$\left[F - \sum_{i=1}^{n} \lambda_i x_i^{\cdot} - \sum_{i,k} \lambda_i^k (\eta_i^k)^{\cdot} - \sum_{i,k} \lambda_{vi}^k (\eta_{vi}^k)^{\cdot} - \sum_{i,k} \lambda_{ai}^k (\eta_{ai}^k)^{\cdot} - \sum_{i,k} \lambda_{pi}^k (\eta_{pi}^k)^{\cdot}\right]_{t_1} = 0 \tag{3.3}$$

$$F^* = F + \sum_i \lambda_i (x_i^{\cdot} - f_i) + \sum_{i,k} \lambda_i^k H_i^k + \sum_{i,k} \lambda_{vi}^k H_{vi}^k + \sum_{i,k} \lambda_{ai}^k H_{ai}^k + \sum_{i,k} \lambda_{pi}^k H_{pi}^k$$

$$\Phi^* = \Phi + \sum_i \nu_i \psi_i \tag{3.4}$$

Here the $\lambda_i, \lambda_i^k, \lambda_{vi}^k, \lambda_{ai}^k, \lambda_{pi}^k$ and ν_j are Lagrange multipliers; the relation (3.3) holds if t_1 is arbitrary.

The necessary optimization conditions together with the original equations of the problem permit finding the optimum, in the mean, control $[u_i(t), a_i]$ and the corresponding programed motion of the system. Then the characteristics of the optimized random process are determined from (1.4) and (1.6).

In conclusion, let us note that by virtue of the reciprocity of the variational problems, the optimization conditions written above permit also the solution of the problems of determining the extremum of some characteristic of the programed motion for a given value of the functional (2.5), i.e. for given accuracy of performing the programed motion.

4. As an example, let us consider a flying vehicle for which the motion of the center of mass in the vertical plane under constant gravity and zero aerodynamic forces is described by the equations

$$X_1^{\cdot} = \frac{WP}{1-Pt}\cos u,\ X_3^{\cdot} = X_2,\ X_2^{\cdot} = \frac{WP}{1-Pt}\sin\ u - g,\quad X_1(0) = X_2(0) = X_3(0) = 0 \tag{4.1}$$

Here X_1 and X_2 are the horizontal and vertical components of the velocity; X_3 is the flight altitude; W the relative velocity of the escaping combustion products of the engine, is a given number; P a random variable with a given mathematical expectation p and variance D characterizes the intensity of fuel consumption; u the angle of slope of the engine thrust to the horizon is not a random control function; g is the acceleration of gravity.

From (4.1) we have the following system to determine the programed motion and deviations from it.

$$x_1^{\cdot} = \frac{Wp}{1-pt}\cos u,\qquad x_2^{\cdot} = \frac{Wp}{1-pt}\sin u - g,\qquad x_3^{\cdot} = x_2$$
$$x_1(0) = x_2(0) = x_3(0) = 0 \tag{4.2}$$

$$X_1^{\circ\cdot} = \frac{WP^\circ}{(1-pt)^2}\cos u,\qquad X_2^{\circ\cdot} = \frac{WP^\circ}{(1-pt)^2}\sin u\ \xi_1\ \xi_2,\qquad X_3^{\circ\cdot} = X_2^\circ$$
$$X_1(0) = X_2(0) = X_3(0) = 0 \tag{4.3}$$

Here $X_i^\circ = X_i - x_i$, $P^\circ = P - p$, x_i is the mathematical expectation of the functions X_i.

The possibility of the following representation of the function X_i°:

$$X_i^\circ = \eta_i P^\circ \qquad (i = 1, 2, 3) \tag{4.4}$$

is seen from (4.3). Here η_1, η_2 and η_3 are determined from the equations

$$\eta_1^{\cdot} = \frac{W}{(1-pt)^2}\cos u,\qquad \eta_2^{\cdot} = \frac{W}{(1-pt)^2}\sin u\ \xi_1\ \xi_2,\qquad \eta_3^{\cdot} = \eta_2$$
$$\eta_1(0) = \eta_2(0) = \eta_3(0) = 0 \tag{4.5}$$

The final values of the functions $X_{i1}^\circ = X_i^\circ(t_1)$ and their variances D_{i1} are written as

$$X_{i1}^\circ = \eta_{i1}P^\circ,\qquad D_{i1} = \eta_{i1}^2 D$$

Let the parameter

$$\Phi = (g_1\eta_{11}^2 + g_2\eta_{21}^2 + g_3\eta_{31}^2)\,D \tag{4.6}$$

where g_1, g_2 and g_3 are non-negative weight constants taken as the measure of the scattering of the motion characteristics at the end of the trajectory. Then considering this parameter to be given, it is possible to solve various problems, optimum in the sense of [9 to 11], under the additional boundary condition (4.6), compliance with which means the determination of optimum flight conditions of a flying vehicle with a scattering level of its final characteristics given in advance.

For example, let us consider the problem of selecting the control $u(t)$ which will guarantee achievement of the maximum horizontal velocity component

$$x_{11} = x_1(t_1) \tag{4.7}$$

for given $x_1(0)$, $x_2(0)$, $x_3(0)$ and t_1 final values of the functions

$$x_2(t_1) = x_{21},\qquad x_3(t_1) = x_{31} \tag{4.8}$$

and a given scattering parameter Φ .

This problem will be a variational problem of Mayer type to seek the functions $u, x_1, x_2, x_3, \eta_1, \eta_2$ and η_3, satisfying Equations (4.2), (4.5), (4.6) and (4.8), and creating the maximum for the functional (4.7). Its solution leads to the following expressions determining the optimum control

$$\tan u = \frac{p\lambda_2 (1 - pt) + \mu_2}{p\lambda_1 (1 - pt) + \mu_1}$$

$$\lambda_1 = -1, \qquad \mu_1 = -2\nu D g_1 \eta_{11}, \qquad \mu_3 = -2\nu D g_3 \eta_{31}$$

$$\lambda_2 = C_1 + C_2 (t_1 - t), \qquad \mu_2 = \mu_{21} + \mu_3 (t_1 - t), \qquad \mu_{21} = -2\nu D g_2 \eta_{21}$$

Here $\lambda_1, \lambda_2, \mu_1, \mu_2, \mu_3$ and ν are Lagrange multipliers.

For comparison, let us note that if the problem is solved without the constraint (4.6), the expression for $\tan u$ is obtained as a linear function of time.

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